Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Displacements depending on One, Two and Three Parameters in a Space of Four Dimensions.

By Thomas Craig.

In the present paper I have given as briefly as possible a generalization to a space of four dimensions of the kinematical methods developed by Darboux in the first two volumes of his "Théorie générale des Surfaces." As I have only a very slight knowledge of what has been done in the geometry of a four-dimensional space (Euclidian), I have confined myself entirely to the generalization of Darboux's formulæ, lest otherwise I might merely repeat what is already well known.

We shall first consider a system having one point, O, fixed. Let X, Y, Z, W be the coordinates of a point referred to fixed rectangular axes having O as origin; x, y, z, w the coordinates of the same point referred to moving axes also having O as origin. The following table gives the direction cosines of the two sets of axes referred one to the other:

$$\left.\begin{array}{c|cccc} & X & Y & Z & W \\ \hline x & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ y & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ z & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ w & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{array}\right\} \qquad (1)$$

Among these 16 cosines we have the following 10 relations:

$$\left.\begin{array}{ll} \alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2=1, & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2+\delta_1\delta_2=0, \\ \alpha_2^2+\beta_2^2+\gamma_2^2+\delta_2^2=1, & \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3+\delta_1\delta_3=0, \\ \alpha_3^2+\beta_3^2+\gamma_3^2+\delta_3^2=1, & \alpha_1\alpha_4+\beta_1\beta_4+\gamma_1\gamma_4+\delta_1\delta_4=0, \\ \alpha_4^2+\beta_4^2+\gamma_4^2+\delta_4^2=1, & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3+\delta_2\delta_3=0, \\ & \alpha_2\alpha_4+\beta_2\beta_4+\gamma_2\gamma_4+\delta_2\delta_4=0, \\ & \alpha_3\alpha_4+\beta_3\beta_4+\gamma_3\gamma_4+\delta_3\delta_4=0, \end{array}\right\} \qquad (2)$$

or

$$\left.\begin{array}{ll} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1, & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = 0, \\ \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1, & \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \alpha_4\gamma_4 = 0, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = 1, & \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3 + \gamma_4\delta_4 = 0, \end{array}\right\} \tag{2'}$$

I shall call the figure formed by the four three-dimensional spaces

$$X = 0, \quad Y = 0, \quad Z = 0, \quad W = 0, \tag{3}$$

meeting at the point O a *tetrahedroid*, and that formed by the six planes

$$\left.\begin{array}{lll} \begin{cases} X = 0, \\ Y = 0, \end{cases} & \begin{cases} X = 0, \\ Z = 0, \end{cases} & \begin{cases} X = 0, \\ W = 0, \end{cases} \\ \begin{cases} Y = 0, \\ Z = 0, \end{cases} & \begin{cases} Y = 0, \\ W = 0, \end{cases} & \begin{cases} Z = 0, \\ W = 0, \end{cases} \end{array}\right\} \tag{4}$$

a *hexahedron*.

It will be convenient to speak of any such expression as

$$AX + BY + CZ + DW + E = 0$$

or

$$ax + by + cz + dw + e = 0$$

as a *hyperplane*.

In fact I shall employ the nomenclature used by Poincaré in his memoir "Sur les Résidus des Intégrales doubles," Acta Math., t. 9, pg. 325; thus a *hypersurface* will be expressed by a single relation between the four coordinates of a point, a *surface* by two such relations, a line by three relations.

The following four equations

$$\begin{array}{ll} a & \alpha_1 x + \beta_1 x + \gamma_1 z + \delta_1 w = 0, \\ b & \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 w = 0, \\ c & \alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3 w = 0, \\ d & \alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4 w = 0 \end{array} \tag{5}$$

are the equations of the four hyperplanes forming the faces of the moving tetrahedroid. The planes formed by the combination of these in pairs will be the faces of the hexahedron; these may be denoted as follows:

$$\left.\begin{array}{lll} (a, b) = P^1, & (a, c) = P^2, & (a, d) = P^3, \\ (b, c) = P^4, & (b, d) = P^5, & (c, d) = P^6. \end{array}\right\} \tag{6}$$

To denote the direction cosines of these six planes I employ the notation used by Cole in his paper "On Rotations in Space of Four Dimensions," American

Journal of Mathematics, vol. XII, p. 191. The direction cosines of the plane P^1, for example, will be denoted by

$$P^1_{12},\quad P^1_{13},\quad P^1_{14},\quad P^1_{23},\quad P^1_{24},\quad P^1_{34}.$$

Since our system is an orthogonal one, we have

$$\left.\begin{aligned} P^1_{12} &= \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}, \quad P^1_{13} = \begin{vmatrix} \alpha_1, & \gamma_1 \\ \alpha_2, & \gamma_2 \end{vmatrix}, \quad P^1_{14} = \begin{vmatrix} \alpha_1, & \delta_1 \\ \alpha_2, & \delta_2 \end{vmatrix}, \\ P^1_{23} &= \begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, \quad P^1_{24} = \begin{vmatrix} \beta_1, & \delta_1 \\ \beta_2, & \delta_2 \end{vmatrix}, \quad P^1_{34} = \begin{vmatrix} \gamma_1, & \delta_1 \\ \gamma_2, & \delta_2 \end{vmatrix}, \end{aligned}\right\} \tag{7}$$

and similar expressions for the 30 direction cosines of the remaining five planes. These quantities P_{ij} satisfy the two equations

$$\left.\begin{aligned} P^2_{12} + P^2_{13} + P^2_{14} + P^2_{23} + P^2_{24} + P^2_{34} &= 1, \\ P_{12}\,P_{34} - P_{13}\,P_{24} + P_{23}\,P_{14} &= 0 \end{aligned}\right\} \tag{8}$$

(note that $P_{24} = - P_{42}$, so that the second of these can be written in the form

$$P_{12}P_{34} + P_{13}P_{42} + P_{23}P_{14} = 0,$$

the form in which Cole gives it).

I shall denote the moving tetrahedroid by the letter T and the hexahedron by H. To form the equations giving the motion of T we proceed in exactly the same manner as in forming the analogous kinematical equations for space of three dimensions.

Differentiate the second set of equations (2′) and write

$$\begin{aligned} \sum\beta_i\frac{d\alpha_i}{dt} &= -\sum\alpha_i\frac{d\beta_i}{dt} = p_{12}, \\ \sum\alpha_i\frac{d\gamma_i}{dt} &= -\sum\gamma_i\frac{d\alpha_i}{dt} = p_{13}, \\ \sum\gamma_i\frac{d\beta_i}{dt} &= -\sum\beta_i\frac{d\gamma_i}{dt} = p_{23}, \\ \sum\delta_i\frac{d\alpha_i}{dt} &= -\sum\alpha_i\frac{d\delta_i}{dt} = p_{14}, \\ \sum\delta_i\frac{d\beta_i}{dt} &= -\sum\beta_i\frac{d\delta_i}{dt} = p_{24}, \\ \sum\delta_i\frac{d\gamma_i}{dt} &= -\sum\gamma_i\frac{d\delta_i}{dt} = p_{34}, \qquad i = 1,\ 2,\ 3,\ 4. \end{aligned}$$

For $w=0$ the quantities p_{23}, p_{13}, p_{12} are respectively the quantities p, q, r of ordinary space. The extended form of the ordinary kinematical equations can now be written down at once, but instead of writing them for the point (x, y, z, w) I shall at once write them for a point at distance unity from O on each of the axes X, Y, Z, W, referring the reader to Darboux ("Théorie générale des Surfaces," t. I, pg. 4) for the intermediate steps. The equations are

$$\begin{aligned}
\frac{d\alpha}{dt} &= \phantom{-p_{12}\alpha} + p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\
\frac{d\beta}{dt} &= -p_{12}\alpha \phantom{- p_{23}\beta} + p_{23}\gamma + p_{24}\delta, \\
\frac{d\gamma}{dt} &= p_{13}\alpha - p_{23}\beta \phantom{+p_{23}\gamma} + p_{34}\delta, \\
\frac{d\delta}{dt} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma
\end{aligned} \tag{10}$$

These equations are satisfied by the four sets of direction cosines

$$(\alpha_1, \beta_1, \gamma_1, \delta_1), \quad (\alpha_2, \beta_2, \gamma_2, \delta_2), \quad (\alpha_3, \beta_3, \gamma_3, \delta_3), \quad (\alpha_4, \beta_4, \gamma_4, \delta_4),$$

and these by (2) can be expressed in terms of 6 arbitrary quantities. For the expression of the 16 cosines in terms of 6 arbitrary quantities the reader is referred to a paper by Cayley in vol. XXXII of Crelle and to Cole's paper above cited.

We have here what at first sight seems rather curious, viz. a system of *four* equations of the first order (equations (10)) containing *six* arbitrary constants in their general solution. This is, however, easy to explain. The rotations p_{ij} are the components of rotation about the six planes of the hexahedron H. By aid of (7) and (10) we can form *six* equations giving the derivatives of $P_{12}, P_{13} \ldots . P_{34}$ with respect to t. We have by (7)

$$\frac{dP^1_{12}}{dt} = \frac{d}{dt}(\alpha_1\beta_2 - \alpha_2\beta_1); \tag{11}$$

expanding this and substituting from (10) where we give the quantities $\alpha, \beta, \gamma, \delta$ the suffixes 1 and 2 successively, we arrive at the equation

$$\frac{dP^1_{12}}{dt} = p_{23}P^1_{13} + p_{24}P^1_{14} + p_{13}P^1_{23} - p_{14}P^1_{24}, \tag{12}$$

and similar equations for

$$\frac{dP^1_{13}}{dt}, \quad \frac{dP^1_{14}}{dt}, \quad \frac{dP^1_{23}}{dt}, \quad \frac{dP^1_{24}}{dt}, \quad \frac{dP^1_{34}}{dt}.$$

Dropping the superior affix, we have the following six equations satisfied by the 36 direction cosines P_{ij}^{k}:

$$\left.\begin{aligned}
\frac{dP_{12}}{dt} &= p_{23}P_{13} + p_{24}P_{14} + p_{13}P_{23} - p_{14}P_{24} \\
\frac{dP_{13}}{dt} &= -p_{23}P_{12} + p_{34}P_{14} + p_{12}P_{23} - p_{14}P_{34}, \\
\frac{dP_{14}}{dt} &= -p_{24}P_{12} - p_{34}P_{13} + p_{12}P_{24} - p_{13}P_{34}, \\
\frac{dP_{23}}{dt} &= -p_{13}P_{12} - p_{12}P_{13} + p_{34}P_{24} - p_{24}P_{34}, \\
\frac{dP_{24}}{dt} &= p_{14}P_{12} + p_{12}P_{14} - p_{34}P_{23} + p_{23}P_{34}, \\
\frac{dP_{34}}{dt} &= p_{14}P_{13} + p_{13}P_{14} + p_{24}P_{23} - p_{23}P_{24} \quad .
\end{aligned}\right\} \qquad (13)$$

From these we have at once the two quadratic integrals

$$\left.\begin{aligned}
P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2 &= \text{const.}, \\
P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} &= \text{const.}
\end{aligned}\right\} \qquad (14)$$

These last equations show us that by proper substitutions the integration of equations (13) can be conducted to the integration of a system of four equations. The four equations are evidently equations (10).

Equations (10) obviously have the following integral of the second degree:

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \text{const.} \qquad (15)$$

If now $(\alpha^0, \beta^0, \gamma^0, \delta^0)$ is a particular integral of (10), we can add the integral of the first degree

$$\alpha\alpha^0 + \beta\beta^0 + \gamma\gamma^0 + \delta\delta^0 = \text{const.} \qquad (16)$$

to (15) (Darboux, I, p. 20); and so if we have three particular integrals of (10)

$$(\alpha^0, \beta^0, \gamma^0, \delta^0), \quad (\alpha^1, \beta^1, \gamma^1, \delta^1), \quad (\alpha^2, \beta^2, \gamma^2, \delta^2),$$

we have the following system of equations for the determination of the general integral:

$$\left.\begin{aligned}
\alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= \text{const.}, \\
\alpha\alpha^0 + \beta\beta^0 + \gamma\gamma^0 + \delta\delta^0 &= \text{const.}, \\
\alpha\alpha^1 + \beta\beta^1 + \gamma\gamma^1 + \delta\delta^1 &= \text{const.}, \\
\alpha\alpha^2 + \beta\beta^2 + \gamma\gamma^2 + \delta\delta^2 &= \text{const.},
\end{aligned}\right\} \qquad (17)$$

Since
$$\alpha^{k^2} + \beta^{k^2} + \gamma^{k^2} + \delta^{k^2} = \text{const.}, \qquad (k = 0 , 1 , 2) \tag{18}$$

we can join to equations (17) the following:[*]

$$\Delta = \begin{vmatrix} \alpha , & \beta , & \gamma , & \delta \\ \alpha^0, & \beta^0, & \gamma^0, & \delta^0 \\ \alpha^1, & \beta^1, & \gamma^1, & \delta^1 \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2 \end{vmatrix} = \text{const.} \tag{19}$$

That $\Delta = \text{const.}$ is obvious from the fact that by (15), (17) and (18) its square is a constant. The last three equations of (17) and equation (19) are linear in α, β, γ, δ, and so serve to give us the values of these four quantities.

It is only necessary to employ the reasoning on page 6 of Darboux, t. I, and equations (10), (15) and (16), to see that a general solution of (10) involves six arbitrary constants, viz. the six arbitrary constants which serve to define the initial position of the hexahedron H.

Suppose now that the system has no fixed point. We must then introduce the components
$$\xi , \ \eta , \ \zeta , \ \tau$$
of translation of the origin of the moving axes. Here again it is only necessary to reproduce Darboux's reasoning (p. 7 *loc. cit.*) Let X_0, Y_0, Z_0, W_0 denote the coordinates of the moving origin. So far as the *rotation* is concerned, the origin can be fixed, and therefore the 16 direction cosines determined as above. We have now the following equations for the determination of X_0, Y_0, Z_0, W_0:

$$\left.\begin{aligned} \frac{dX_0}{dt} &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta + \delta_1\tau , \\ \frac{dY_0}{dt} &= \alpha_2\xi + \beta_2\eta + \gamma_2\zeta + \delta_2\tau , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dW_0}{dt} &= \alpha_4\xi + \beta_4\eta + \gamma_4\zeta + \delta_4\tau , \end{aligned}\right\} \tag{20}$$

---

[*] This method of solving equations (17) was, for three dimensions, communicated to me by Professor Echols of the University of Virginia, who had received it from M. E. Cosserat. M. Cosserat's note to Professor Echols was drawn out by a solution which I had published in the Annals of Mathematics of the equations just preceding (6) on page 21 of Darboux's Théorie générale des Surfaces.

The integration of these equations is reduced to quadratures, since the coefficients α_i, β_i, γ_i, δ_i have already been determined as functions of t and ξ, η, ζ, τ, like $p_{12}, p_{13} \ldots . p_{34}$, are *given* functions of t.

Return now to equations (10) and write

$$\alpha = \frac{2\lambda}{k^2+1}, \quad \beta = \frac{2\mu}{k^2+1}, \quad \gamma = \frac{2\nu}{k^2+1}, \quad \delta = \frac{k^2-1}{k^2+1}, \tag{21}$$

$$k^2 = \lambda^2 + \mu^2 + \nu^2. \tag{22}$$

After some simple reductions we shall find the following equations:

$$\left.\begin{aligned} \frac{d\lambda}{dt} &= \mu p_{12} - \nu p_{13} + \left(\frac{k^2-1}{2}\right) p_{14} - \lambda\,[\lambda p_{14} + \mu p_{24} + \nu p_{34}], \\ \frac{d\mu}{dt} &= -\lambda p_{12} + \nu p_{23} + \left(\frac{k^2-1}{2}\right) p_{24} - \mu\,[\lambda p_{14} + \mu p_{24} + \nu p_{34}], \\ \frac{d\nu}{dt} &= \lambda p_{13} - \mu p_{23} + \left(\frac{k^2-1}{2}\right) p_{34} - \nu\,[\lambda p_{14} + \mu p_{24} + \nu p_{34}]. \end{aligned}\right\} \tag{23}$$

This system of simultaneous equations is, for three unknown functions, a generalization of Riccati's equation. I do not know whether such equations have been studied or not, but, as their integration plays no part in the present paper, it is not necessary to say anything more about them.[*]

Consider now the case of displacements depending on two parameters, t, u. Let $p_{12}, p_{13}, \ldots . p_{34}$ denote the rotations which depend on t alone; $p'_{12}, p'_{13}, \ldots . p'_{34}$ those which depend on u alone. We have at once the equations

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial t} &= \phantom{-p_{12}\alpha} \quad p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\ \frac{\partial\beta}{\partial t} &= -p_{12}\alpha \phantom{- p_{13}\gamma} + p_{23}\gamma + p_{24}\delta, \\ \frac{\partial\gamma}{\partial t} &= p_{13}\alpha - p_{23}\beta \phantom{+ p_{23}\gamma} + p_{34}\delta, \\ \frac{\partial\delta}{\partial t} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma \qquad , \end{aligned}\right\} \tag{24}$$

---

[*] A note on the subject of these equations by Mr. John Eiesland will appear in the following number of this Journal.

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial u} &= \qquad\qquad p'_{12}\beta - p'_{13}\gamma + p'_{14}\delta,\\
\frac{\partial\beta}{\partial u} &= -p'_{12}\alpha \qquad\quad + p'_{23}\gamma + p'_{24}\delta,\\
\frac{\partial\gamma}{\partial u} &= \quad p'_{13}\alpha - p'_{23}\beta \qquad\quad + p'_{34}\delta,\\
\frac{\partial\delta}{\partial u} &= -p'_{14}\alpha - p'_{24}\beta - p'_{34}\gamma \qquad .
\end{aligned}\right\} \tag{25}$$

Differentiating each of (24) for u and each of (25) for v and equating the results, we have the following six equations of condition connecting the coefficients p_{ij} and p'_{ij}:

$$\left.\begin{aligned}
\frac{\partial p_{12}}{\partial u} - \frac{\partial p'_{12}}{\partial t} + \begin{vmatrix} p_{13}, & p_{23} \\ p'_{13}, & p'_{23} \end{vmatrix} - \begin{vmatrix} p_{14}, & p_{24} \\ p'_{14}, & p'_{24} \end{vmatrix} &= 0,\\
\frac{\partial p_{13}}{\partial u} - \frac{\partial p'_{13}}{\partial t} + \begin{vmatrix} p_{23}, & p_{12} \\ p'_{23}, & p'_{12} \end{vmatrix} - \begin{vmatrix} p_{34}, & p_{14} \\ p'_{34}, & p'_{14} \end{vmatrix} &= 0,\\
\frac{\partial p_{14}}{\partial u} - \frac{\partial p'_{14}}{\partial t} + \begin{vmatrix} p_{12}, & p_{24} \\ p'_{1}\ , & p'_{24} \end{vmatrix} - \begin{vmatrix} p_{13}, & p_{34} \\ p'_{13}, & p'_{34} \end{vmatrix} &= 0,\\
\frac{\partial p_{23}}{\partial u} - \frac{\partial p'_{23}}{\partial t} + \begin{vmatrix} p_{12}, & p_{13} \\ p'_{12}, & p'_{13} \end{vmatrix} - \begin{vmatrix} p_{24}, & p_{34} \\ p'_{24}, & p'_{34} \end{vmatrix} &= 0,\\
\frac{\partial p_{24}}{\partial u} - \frac{\partial p'_{24}}{\partial t} + \begin{vmatrix} p_{14}, & p_{12} \\ p'_{14}, & p'_{12} \end{vmatrix} - \begin{vmatrix} p_{34}, & p_{23} \\ p'_{34}, & p'_{23} \end{vmatrix} &= 0,\\
\frac{\partial p_{34}}{\partial u} - \frac{\partial p'_{34}}{\partial t} + \begin{vmatrix} p_{13}, & p_{14} \\ p'_{13}, & p'_{14} \end{vmatrix} - \begin{vmatrix} p_{23}, & p_{24} \\ p'_{23}, & p'_{24} \end{vmatrix} &= 0.
\end{aligned}\right\} \tag{26}$$

We shall arrive at the same set of relations between the quantities p_{ij} and p'_{ij} if we form the two systems of equations similar to (13), one system giving the values of $\frac{\partial P_{ij}}{\partial t}$ and the other giving the values of $\frac{\partial P_{ij}}{\partial u}$. For convenience of reference I shall write the first of each of these systems:

$$\left.\begin{aligned}
\frac{\partial P_{12}}{\partial t} &= p_{23}P_{13} + p_{24}P_{14} + p_{13}P_{23} - p_{14}P_{24},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned}
\frac{\partial P_{12}}{\partial u} &= p'_{23}P_{13} + p'_{24}P_{14} + p'_{13}P_{23} - p'_{14}P_{24},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{28}$$

Reciprocally, whenever we have twelve quantities p_{ij}, p'_{ij} satisfying equations (26), there exists a motion in which these twelve quantities are the rotations.

We can use either equations (24) and (25) or (27) and (28) and reproduce almost word for word the reasoning on pages 49–51 of Darboux;* it is not necessary to go over this ground, as the reader can readily supply the missing reasoning.

The question of the integration is in this case led back to the determination of systems of solutions common to two sets of equations formed like (23)—one gives

$$\frac{\partial x}{\partial t}, \quad \frac{\partial y}{\partial t}, \quad \frac{\partial z}{\partial t},$$

and the other

$$\frac{\partial x}{\partial u}, \quad \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial u}.$$

It is not the purpose of the present paper to take up the study of any of these equations, but merely to indicate what problems require to be solved in generalizing Darboux's methods. If the system has no fixed point, let ξ, η, ζ, τ denote the translations of the origin of T, or H, depending only on t and ξ', η', ζ', τ', the translations depending only on u; if X_0, Y_0, Z_0, W_0 denote the coordinates of the moving origin, we have for their determination the equations

$$\left.\begin{aligned} \frac{\partial X_0}{\partial t} &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta + \delta_1\tau , \\ \frac{\partial X_0}{\partial u} &= \alpha_1\xi' + \beta_1\eta' + \gamma_1\zeta' + \delta_1\tau', \end{aligned}\right\} \qquad (29)$$

with similar expressions in Y_0, Z_0, W_0. The 16 cosines are, of course, determined just as in the case where the system had one fixed point. To find the conditions to be satisfied by ξ, η, ... τ', differentiate the first of (29) for u and the second for t and equate the results. Since the equations must hold when we replace $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ by $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ or $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ or $(\alpha_4, \beta_4, \gamma_4, \delta_4)$, and since the determinant formed by the 16 cosines is not zero, we have at once the following equations of condition:

$$\left.\begin{aligned} \frac{\partial \xi}{\partial u} - \frac{\partial \xi'}{\partial t} + p_{12}\eta' - p'_{12}\eta - p_{13}\zeta' + p'_{13}\zeta + p_{14}\tau' - p'_{14}\tau &= 0, \\ \frac{\partial \eta}{\partial u} - \frac{\partial \eta'}{\partial t} - p_{12}\xi' + p'_{12}\xi + p_{23}\zeta' - p'_{23}\zeta + p_{24}\tau' - p'_{24}\tau &= 0, \\ \frac{\partial \zeta}{\partial u} - \frac{\partial \zeta'}{\partial t} + p_{13}\xi' - p'_{13}\xi - p_{23}\eta' + p'_{23}\eta + p_{34}\tau' - p'_{34}\tau &= 0, \\ \frac{\partial \tau}{\partial u} - \frac{\partial \tau'}{\partial t} - p_{14}\xi' + p'_{14}\xi - p_{24}\eta' + p'_{24}\eta - p_{34}\tau' + p'_{34}\tau &= 0, \end{aligned}\right\} \qquad (30)$$

---

*It is understood that the references are always to t. I of the "Théorie générale des Surfaces."

Reciprocally, whenever the quantities $\xi \ldots . \tau'$, $p_{12} \ldots . p'_{34}$ satisfy equations (26) and (30), there exists a motion for which these are the translations and rotations; to show this it is only necessary to reproduce the reasoning on page 67 of Darboux; this, however, will not be done.

The following are the projections of the infinitesimal displacements on the axes of x, y, z, w in the case of no fixed point; and when the variables (t, u) are supposed to depend on a single parameter, say s:

$$\left.\begin{array}{l} dx + \xi dt + \xi' du + (p_{12}dt + p'_{12}du)\, y - (p_{13}dt + p'_{13}du)\, z + (p_{14}dt + p'_{14}du)\, w\,, \\ dy + \eta dt + \eta' du - (p_{12}dt + p'_{12}du)\, x + (p_{23}dt + p'_{23}du)\, z + (p_{24}dt + p'_{24}du)\, w\,, \\ dz + \zeta dt + \zeta' du + (p_{13}dt + p'_{13}du)\, x - (p_{23}dt + p'_{23}du)\, y + (p_{34}dt + p'_{34}du)\, w, \\ dw + \tau dt + \tau' du - (p_{14}dt + p'_{14}du)\, x - (p_{24}dt + p'_{24}du)\, y - (p_{34}dt + p'_{34}du)\, z\,. \end{array}\right\} \quad (31)$$

For the case of a fixed point it is only necessary to make all of the Greek letters zero, and for a one-variable displacement to make the quantities p'_{ij} all zero.

In the case of a one-variable displacement and a curve of triple curvature, the geometrical interpretation of the quantities p_{ij} is given in a Thesis by Mr. J. G. Hardy which will shortly be published. There also will be found a fuller account of some matters which I have merely indicated in what precedes and follows.

In the case of a one-variable displacement we are of course conducted to the geometry of a curve in 4-dimensional space, that is, a locus represented by three equations in x, y, z, w, say

$$\left.\begin{array}{l} \phi_1(x, y, z, w) = 0\,, \\ \phi_2(x, y, z, w) = 0\,, \\ \phi_3(x, y, z, w) = 0\,; \end{array}\right\} \quad (32)$$

in the case of two-variable displacements we are conducted to the geometry of a surface which may be denoted by the two equations

$$\left.\begin{array}{l} \psi_1(x, y, z, w) = 0\,, \\ \psi_2(x, y, z, w) = 0\,. \end{array}\right\} \quad (33)$$

We can, however, go a step further and consider three-variable displacements when a point moves on (or in?) a curved 3-dimensional space or briefly on a hypersurface. Let t, u, v denote the independent displacement variables in this case. We shall now have the following three sets of equations:

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial t} &= \phantom{-p_{12}\alpha+{}} p_{12}\beta - p_{13}\gamma + p_{14}\delta,\\
\frac{\partial\beta}{\partial t} &= -p_{12}\alpha \phantom{-p_{23}\beta} + p_{23}\gamma + p_{24}\delta,\\
\frac{\partial\gamma}{\partial t} &= p_{13}\alpha - p_{23}\beta \phantom{+p_{23}\gamma} + p_{34}\delta,\\
\frac{\partial\delta}{\partial t} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma \phantom{+p_{34}\delta},
\end{aligned}\right\}\tag{34}$$

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial u} &= \phantom{-p'_{12}\alpha+{}} p'_{12}\beta - p'_{13}\gamma + p'_{14}\delta,\\
\frac{\partial\beta}{\partial u} &= -p'_{12}\alpha \phantom{-p'_{23}\beta} + p'_{23}\gamma + p'_{24}\delta,\\
\frac{\partial\gamma}{\partial u} &= p'_{13}\alpha - p'_{23}\beta \phantom{+p'_{23}\gamma} + p'_{34}\delta,\\
\frac{\partial\delta}{\partial u} &= -p'_{14}\alpha - p'_{24}\beta - p'_{34}\gamma \phantom{+p'_{34}\delta},
\end{aligned}\right\}\tag{34'}$$

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial v} &= \phantom{-p''_{12}\alpha+{}} p''_{12}\beta - p''_{13}\gamma + p''_{14}\delta,\\
\frac{\partial\beta}{\partial v} &= -p''_{12}\alpha \phantom{-p''_{23}\beta} + p''_{23}\gamma + p''_{24}\delta,\\
\frac{\partial\gamma}{\partial v} &= p''_{13}\alpha - p''_{23}\beta \phantom{+p''_{23}\gamma} + p''_{34}\delta,\\
\frac{\partial\delta}{\partial v} &= -p''_{14}\alpha - p''_{24}\beta - p''_{34}\gamma \phantom{+p''_{34}\delta}.
\end{aligned}\right\}\tag{34''}$$

Three other sets of equations similar to (13) and giving the values of

$$\frac{\partial P_{ij}}{\partial t},\quad \frac{\partial P_{ij}}{\partial u},\quad \frac{\partial P_{ij}}{\partial v}$$

can also be written down, but their forms are so obvious that it is not worth while taking up space by reproducing them. We have now to find the conditions which must exist among the quantities p, p' and p'' in order that the three preceding systems of equations may admit of common solutions. These conditions are obtained in the same way as above and are eighteen in number, grouping themselves naturally in six groups of three each, viz.

$$\left.\begin{aligned}
\frac{\partial p_{12}}{\partial u} - \frac{\partial p'_{12}}{\partial t} + \begin{vmatrix} p_{13}, & p_{23} \\ p'_{13}, & p'_{23} \end{vmatrix} - \begin{vmatrix} p_{14}, & p_{24} \\ p'_{14}, & p'_{24} \end{vmatrix} &= 0,\\
\frac{\partial p'_{12}}{\partial v} - \frac{\partial p''_{12}}{\partial u} + \begin{vmatrix} p'_{12}, & p'_{23} \\ p''_{12}, & p''_{23} \end{vmatrix} - \begin{vmatrix} p'_{14}, & p'_{24} \\ p''_{14}, & p''_{24} \end{vmatrix} &= 0,\\
\frac{\partial p''_{12}}{\partial t} - \frac{\partial p_{12}}{\partial v} + \begin{vmatrix} p_{23}, & p_{13} \\ p''_{23}, & p''_{13} \end{vmatrix} - \begin{vmatrix} p_{24}, & p_{14} \\ p'_{24}, & p'_{14} \end{vmatrix} &= 0,
\end{aligned}\right\}\tag{A_{12}}$$

$$\left.\begin{aligned}
&\frac{\partial p_{13}}{\partial u}-\frac{\partial p'_{13}}{\partial t}+\begin{vmatrix} p_{22}, & p_{12} \\ p'_{23}, & p_{12} \end{vmatrix}-\begin{vmatrix} p_{34}, & p_{14} \\ p'_{34}, & p_{14} \end{vmatrix}=0, \\
&\frac{\partial p'_{13}}{\partial t}-\frac{\partial p''_{13}}{\partial u}+\begin{vmatrix} p'_{23}, & p'_{12} \\ p''_{23}, & p''_{12} \end{vmatrix}-\begin{vmatrix} p'_{34}, & p'_{1} \\ p''_{34}, & p'_{14} \end{vmatrix}=0, \\
&\frac{\partial p''_{13}}{\partial t}-\frac{\partial p_{13}}{\partial v}+\begin{vmatrix} p''_{23}, & p''_{12} \\ p_{23}, & p_{12} \end{vmatrix}-\begin{vmatrix} p''_{34}, & p''_{14} \\ p_{34}, & p_{14} \end{vmatrix}=0,
\end{aligned}\right\}\qquad (\mathrm{A}_{13})$$

$$\left.\begin{aligned}
&\frac{\partial p_{14}}{\partial u}-\frac{\partial p'_{14}}{\partial t}+\begin{vmatrix} p_{12}, & p_{24} \\ p'_{12}, & p'_{24} \end{vmatrix}-\begin{vmatrix} p_{13}, & p_{34} \\ p'_{13}, & p'_{34} \end{vmatrix}=0, \\
&\frac{\partial p'_{14}}{\partial v}-\frac{\partial p''_{14}}{\partial u}+\begin{vmatrix} p'_{12}, & p'_{24} \\ p''_{14}, & p''_{24} \end{vmatrix}-\begin{vmatrix} p'_{13}, & p'_{34} \\ p''_{13}, & p''_{34} \end{vmatrix}=0, \\
&\frac{\partial p''_{14}}{\partial t}-\frac{\partial p_{14}}{\partial v}+\begin{vmatrix} p''_{12}, & p''_{34} \\ p_{14}, & p_{24} \end{vmatrix}-\begin{vmatrix} p''_{13}, & p''_{34} \\ p_{13}, & p_{34} \end{vmatrix}=0.
\end{aligned}\right\}\qquad (\mathrm{A}_{14})$$

$$\left.\begin{aligned}
&\frac{\partial p_{23}}{\partial u}-\frac{\partial p'_{23}}{\partial t}+\begin{vmatrix} p_{12}, & p_{13} \\ p'_{12}, & p'_{13} \end{vmatrix}-\begin{vmatrix} p_{24}, & p_{34} \\ p'_{24}, & p'_{34} \end{vmatrix}=0, \\
&\frac{\partial p'_{23}}{\partial v}-\frac{\partial p''_{23}}{\partial u}+\begin{vmatrix} p'_{12}, & p'_{13} \\ p''_{12}, & p''_{13} \end{vmatrix}-\begin{vmatrix} p'_{24}, & p'_{34} \\ p''_{24}, & p''_{34} \end{vmatrix}=0, \\
&\frac{\partial p''_{23}}{\partial t}-\frac{\partial p_{23}}{\partial v}+\begin{vmatrix} p''_{12}, & p''_{13} \\ p_{12}, & p_{13} \end{vmatrix}-\begin{vmatrix} p''_{24}, & p''_{34} \\ p_{24}, & p_{34} \end{vmatrix}=0,
\end{aligned}\right\}\qquad (\mathrm{A}_{23})$$

$$\left.\begin{aligned}
&\frac{\partial p_{24}}{\partial u}-\frac{\partial p'_{24}}{\partial t}+\begin{vmatrix} p_{14}, & p_{12} \\ p'_{14}, & p'_{12} \end{vmatrix}-\begin{vmatrix} p_{24}, & p_{23} \\ p'_{24}, & p'_{23} \end{vmatrix}=0, \\
&\frac{\partial p'_{24}}{\partial v}-\frac{\partial p''_{24}}{\partial u}+\begin{vmatrix} p'_{14}, & p'_{12} \\ p''_{14}, & p''_{12} \end{vmatrix}-\begin{vmatrix} p'_{24}, & p'_{23} \\ p''_{24}, & p''_{23} \end{vmatrix}=0, \\
&\frac{\partial p''_{24}}{\partial t}-\frac{\partial p_{24}}{\partial v}+\begin{vmatrix} p''_{14}, & p''_{12} \\ p_{14}, & p_{12} \end{vmatrix}-\begin{vmatrix} p''_{24}, & p''_{23} \\ p_{24}, & p_{23} \end{vmatrix}=0,
\end{aligned}\right\}\qquad (\mathrm{A}_{24})$$

$$\left.\begin{aligned}
&\frac{\partial p_{34}}{\partial u}-\frac{\partial p'_{34}}{\partial t}+\begin{vmatrix} p_{13}, & p_{14} \\ p'_{13}, & p'_{14} \end{vmatrix}-\begin{vmatrix} p_{23}, & p_{24} \\ p'_{23}, & p'_{24} \end{vmatrix}=0, \\
&\frac{\partial p'_{34}}{\partial v}-\frac{\partial p''_{34}}{\partial u}+\begin{vmatrix} p'_{13}, & p'_{14} \\ p''_{13}, & p''_{14} \end{vmatrix}-\begin{vmatrix} p'_{23}, & p'_{24} \\ p''_{23}, & p''_{24} \end{vmatrix}=0, \\
&\frac{\partial p''_{34}}{\partial t}-\frac{\partial p_{34}}{\partial v}+\begin{vmatrix} p''_{13}, & p''_{14} \\ p_{13}, & p_{14} \end{vmatrix}-\begin{vmatrix} p''_{23}, & p''_{24} \\ p_{23}, & p_{24} \end{vmatrix}=0,
\end{aligned}\right\}\qquad (\mathrm{A}_{34})$$

There are several forms into which the three terms in each group can be thrown, many of them rather elegant, but inasmuch as the above are really the most convenient forms for applying these conditions, I shall not give any of them.

A final set of conditions must be obtained when we suppose the system to have no fixed point. Let (ξ, η, ζ, τ) denote the translations of the origin of T depending on t alone; $(\xi', \eta', \zeta', \tau')$ those depending on u alone, and $(\xi'', \eta'', \zeta'', \tau'')$ those depending on v alone. If, as before, in equations (29), X_0, Y_0, Z_0, W_0 denote the coordinates of the moving origin, we shall have

$$\left.\begin{aligned}\frac{\partial X_0}{\partial t} &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta + \delta_1\tau\,,\\ \frac{\partial X_0}{\partial u} &= \alpha_1\xi' + \beta_1\eta' + \gamma_1\zeta' + \delta_1\tau'\,,\\ \frac{\partial X_0}{\partial v} &= \alpha_1\xi'' + \beta_1\eta'' + \gamma_1\zeta'' + \delta_1\tau''.\end{aligned}\right\} \tag{35}$$

The motion of the tetrahedroid is, of course, obtained just as when one point was fixed, so that quadratures only are necessary to determine $X_0, \ldots\ldots W_0$. The conditions to be satisfied by the $\xi, \eta, \ldots\ldots \tau''$ are readily found and are as follows:

$$\begin{aligned}&\frac{\partial\xi}{\partial u} - \frac{\partial\xi'}{\partial t} - \begin{vmatrix}\eta\,, & \eta'\\ p_{12}, & p'_{12}\end{vmatrix} + \begin{vmatrix}\zeta\,, & \zeta'\\ p_{13}, & p'_{13}\end{vmatrix} - \begin{vmatrix}\tau\,, & \tau'\\ p_{14}, & p'_{14}\end{vmatrix} = 0,\\ &\frac{\partial\xi'}{\partial v} - \frac{\partial\xi''}{\partial u} - \begin{vmatrix}\eta'\,, & \eta''\\ p'_{12}, & p''_{12}\end{vmatrix} + \begin{vmatrix}\zeta'\,, & \zeta''\\ p'\,, & p''_{13}\end{vmatrix} - \begin{vmatrix}\tau'\,, & \tau''\\ p'_{14}, & p''_{14}\end{vmatrix} = 0,\\ &\frac{\partial\xi''}{\partial t} - \frac{\partial\xi}{\partial v} - \begin{vmatrix}\eta''\,, & \eta\\ p''_{12}, & p_{12}\end{vmatrix} + \begin{vmatrix}\zeta''\,, & \zeta\\ p''_{13}, & p_{13}\end{vmatrix} - \begin{vmatrix}\tau''\,, & \tau\\ p''_{14}, & p_{14}\end{vmatrix} = 0,\end{aligned} \tag{ξ}$$

$$\begin{aligned}&\frac{\partial\eta}{\partial u} - \frac{\partial\eta'}{\partial t} + \begin{vmatrix}\xi\,, & \xi'\\ p_{12}, & p'_{12}\end{vmatrix} - \begin{vmatrix}\zeta\,, & \zeta'\\ p_{23}, & p'_{23}\end{vmatrix} - \begin{vmatrix}\tau\,, & \tau'\\ p_{24}, & p'_{24}\end{vmatrix} = 0,\\ &\frac{\partial\eta'}{\partial v} - \frac{\partial\eta''}{\partial u} + \begin{vmatrix}\xi'\,, & \xi''\\ p'_{12}, & p''_{12}\end{vmatrix} - \begin{vmatrix}\zeta'\,, & \zeta''\\ p'_{23}, & p''_{23}\end{vmatrix} - \begin{vmatrix}\tau'\,, & \tau''\\ p'_{24}, & p''_{24}\end{vmatrix} = 0,\\ &\frac{\partial\eta''}{\partial t} - \frac{\partial\eta}{\partial v} + \begin{vmatrix}\xi''\,, & \xi\\ p''_{12}, & p_{12}\end{vmatrix} - \begin{vmatrix}\zeta''\,, & \zeta\\ p''_{23}, & p_{23}\end{vmatrix} - \begin{vmatrix}\tau''\,, & \tau\\ p''_{24}, & p_{24}\end{vmatrix} = 0,\end{aligned} \tag{η}$$

$$\begin{aligned}&\frac{\partial\zeta}{\partial u} - \frac{\partial\zeta'}{\partial t} - \begin{vmatrix}\xi\,, & \xi'\\ p_{13}, & p'_{13}\end{vmatrix} + \begin{vmatrix}\eta\,, & \eta'\\ p_{23}, & p'_{23}\end{vmatrix} - \begin{vmatrix}\tau\,, & \tau'\\ p_{34}, & p'_{34}\end{vmatrix} = 0,\\ &\frac{\partial\zeta'}{\partial v} - \frac{\partial\zeta''}{\partial u} - \begin{vmatrix}\xi'\,, & \xi''\\ p'_{13}, & p''_{13}\end{vmatrix} + \begin{vmatrix}\eta'\,, & \eta''\\ p'_{23}, & p''_{23}\end{vmatrix} - \begin{vmatrix}\tau'\,, & \tau''\\ p'_{34}, & p''_{34}\end{vmatrix} = 0,\\ &\frac{\partial\zeta''}{\partial t} - \frac{\partial\zeta}{\partial v} - \begin{vmatrix}\xi''\,, & \xi\\ p''_{13}, & p_{13}\end{vmatrix} + \begin{vmatrix}\eta''\,, & \eta\\ p''_{23}, & p_{23}\end{vmatrix} - \begin{vmatrix}\tau''\,, & \tau\\ p''_{34}, & p_{34}\end{vmatrix} = 0,\end{aligned} \tag{ζ}$$

$$\left.\begin{aligned}
&\frac{\partial\tau}{\partial u}-\frac{\partial\tau'}{\partial t}+\begin{vmatrix}\xi\,, & \xi' \\ p_{14}, & p'_{14}\end{vmatrix}+\begin{vmatrix}\eta\,, & \eta' \\ p_{24}, & p'_{24}\end{vmatrix}+\begin{vmatrix}\zeta\,, & \zeta' \\ p_{34}, & p'_{34}\end{vmatrix}=0,\\
&\frac{\partial\tau'}{\partial v}-\frac{\partial\tau''}{\partial u}+\begin{vmatrix}\xi'\,, & \xi'' \\ p'_{14}, & p''_{14}\end{vmatrix}+\begin{vmatrix}\eta'\,, & \eta'' \\ p'_{24}, & p''_{24}\end{vmatrix}+\begin{vmatrix}\zeta'\,, & \zeta'' \\ p'_{34}, & p''_{34}\end{vmatrix}=0,\\
&\frac{\partial\tau''}{\partial t}-\frac{\partial\tau}{\partial v}+\begin{vmatrix}\xi''\,, & \xi \\ p''_{14}, & p_{14}\end{vmatrix}+\begin{vmatrix}\eta''\,, & \eta \\ p''_{24}, & p_{24}\end{vmatrix}+\begin{vmatrix}\zeta''\,, & \zeta \\ p''_{34}, & p_{34}\end{vmatrix}=0,
\end{aligned}\right. \qquad (\tau)$$

If t, u, v depend on a single parameter, say s, we have for the projections of the infinitesimal displacements on the four axes of x, y, z, w the following:

$$\left.\begin{aligned}
&dx+\xi dt+\xi' du+\xi'' dv+(p_{12}dt+p'_{12}du+p''_{12}dv)\,y\\
&\qquad\qquad -(p_{13}dt+p'_{13}du+p''_{13}dv)\,z+(p_{14}dt+p'_{14}du+p''_{14}dv)\,w,\\
&dy+\eta dt+\eta' du+\eta'' dv-(p_{12}dt+p'_{12}du+p''_{12}dv)\,x\\
&\qquad\qquad +(p_{23}dt+p'_{23}du+p''_{23}dv)\,z+(p_{24}dt+p'_{24}du+p''_{24}dv)\,w,\\
&dz+\zeta dt+\zeta' du+\zeta'' dv+(p_{13}dt+p'_{13}du+p''_{13}dv)\,x\\
&\qquad\qquad -(p_{23}dt+p'_{23}du+p''_{23}dv)\,y+(p_{34}dt+p'_{34}du+p''_{34}dv)\,w,\\
&dw+\tau dt+\tau' du+\tau'' dv-(p_{14}dt+p'_{14}du+p''_{14}dv)\,x\\
&\qquad\qquad -(p_{24}dt+p'_{24}du+p''_{24}dv)\,y-(p_{34}dt+p'_{34}du+p''_{34}dv)\,z.
\end{aligned}\right\} \quad (36)$$

A further hypothesis can be made in this case, viz. that t, u, v depend each on *two* independent parameters, say s and s_1, then we should have

$$\left.\begin{aligned}
dt&=\frac{\partial t}{\partial s}ds+\frac{\partial t}{\partial s_1}ds_1,\\
du&=\frac{\partial u}{\partial s}ds+\frac{\partial u}{\partial s_1}ds_1,\\
dv&=\frac{\partial v}{\partial s}ds+\frac{\partial v}{\partial s_1}ds_1.
\end{aligned}\right\} \qquad (37)$$

In this case the t, u, v can be considered as the coordinates of a point on a surface.

In the case of a three-variable displacement we may assume the origin of T to move on a hypersurface, the axis of w to be normal to the hypersurface, and the other three axes to lie in the tangent hyperplane. We shall then have

$$\tau=\tau'=\tau''=0,$$

and the linear element will then be given by

$$ds^2=(\xi dt+\xi' du+\xi'' dv)^2+(\eta dt+\eta' du+\eta'' dv)^2+(\zeta dt+\zeta' du+\zeta'' dv)^2. \quad (38)$$

Write

$$\left.\begin{aligned} \xi^2 + \eta^2 + \zeta^2 &= E_{11}, \\ \xi'^2 + \eta'^2 + \zeta'^2 &= E_{22}, \\ \xi''^2 + \eta''^2 + \zeta''^2 &= E_{33}, \\ \xi\xi' + \eta\eta' + \zeta\zeta' &= F_{12}, \\ \xi\xi'' + \eta\eta'' + \zeta\zeta'' &= F_{13}, \\ \xi'\xi'' + \eta'\eta'' + \zeta'\zeta'' &= F_{23}, \end{aligned}\right\} \tag{39}$$

then

$$ds^2 = E_{13}dt^2 + E_{23}du^2 + E_{33}dv^2 + 2F_{12}dt\,du + 2F_{13}dt\,dv + 2F_{23}du\,dv. \tag{40}$$

The discriminant of this is

$$\Delta^2 = \begin{vmatrix} E_{11} & F_{12} & F_{13} \\ F_{12} & E_{22} & F_{23} \\ F_{13} & F_{23} & E_{33} \end{vmatrix} = \begin{vmatrix} \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}^2. \tag{41}$$

Of course the values of the E's and F's are known when the hypersurface is given, that is, when we know the values of x, y, z, w in terms of t, u, v; in that case we have

$$\left.\begin{aligned} E_{11} &= \sum\left(\frac{\partial x}{\partial t}\right)^2, & E_{22} &= \sum\left(\frac{\partial x}{\partial u}\right)^2, & E_{33} &= \sum\left(\frac{\partial x}{\partial v}\right)^2, \\ F_{12} &= \sum\frac{\partial x}{\partial t}\,\frac{\partial x}{\partial u}, & F_{13} &= \sum\frac{\partial x}{\partial t}\,\frac{\partial x}{\partial v}, & F_{23} &= \sum\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}. \end{aligned}\right\} \tag{42}$$

Equations (39) and (42) serve to determine $\xi, \eta, \ldots. \zeta''$ so far as they can be determined. Any hypothesis concerning the way in which T is *attached* to the hypersurface will give three other conditions which can be joined to (39). We shall then have nine equations which will serve for the complete determination of $\xi, \eta, \ldots. \zeta''$.

Generalizing now the equation given by Darboux, t. II, p. 376 *et seq.*, we have manifestly

$$\left.\begin{aligned} \xi\alpha_1 + \eta\beta_1 + \zeta\gamma_1 &= \frac{\partial x}{\partial t}, \\ \xi\alpha_2 + \eta\beta_2 + \zeta\gamma_2 &= \frac{\partial y}{\partial t}, \\ \xi\alpha_3 + \eta\beta_3 + \zeta\gamma_3 &= \frac{\partial z}{\partial t}, \\ \xi\alpha_4 + \eta\beta_4 + \zeta\gamma_4 &= \frac{\partial w}{\partial t}, \end{aligned}\right\} \tag{43}$$

$$\left.\begin{array}{l} \xi'\alpha_1 + \eta'\beta_1 + \zeta'\gamma_1 = \dfrac{\partial x}{\partial u}, \\ \dots\dots\dots\dots\dots\dots\dots \\ \xi'\alpha_4 + \eta'\beta_4 + \zeta'\gamma_4 = \dfrac{\partial w}{\partial u}, \end{array}\right\} \qquad (44)$$

$$\left.\begin{array}{l} \xi''\alpha_1 + \eta''\beta_1 + \zeta''\gamma_1 = \dfrac{\partial x}{\partial v}, \\ \dots\dots\dots\dots\dots\dots\dots \\ \xi''\alpha_4 + \eta''\beta_4 + \zeta''\gamma_4 = \dfrac{\partial w}{\partial v}. \end{array}\right\} \qquad (45)$$

From these we find at once:

$$\left.\begin{array}{l} \alpha_1 = \dfrac{1}{\Delta}\begin{vmatrix} \dfrac{\partial x}{\partial t} & \eta & \zeta, \\ \dfrac{\partial x}{\partial u} & \eta' & \zeta', \\ \dfrac{\partial x}{\partial v} & \eta'' & \zeta'', \end{vmatrix} \\ \\ \beta_1 = \dfrac{1}{\Delta}\begin{vmatrix} \xi & \dfrac{\partial x}{\partial t} & \zeta, \\ \xi' & \dfrac{\partial x}{\partial u} & \zeta', \\ \zeta'' & \dfrac{\partial x}{\partial v} & \zeta'', \end{vmatrix} \\ \\ \gamma_1 = \dfrac{1}{\Delta}\begin{vmatrix} \xi & \eta & \dfrac{\partial x}{\partial t}, \\ \xi' & \eta' & \dfrac{\partial x}{\partial u}, \\ \xi'' & \eta'' & \dfrac{\partial x}{\partial v}. \end{vmatrix} \end{array}\right\} \qquad (46)$$

The quantities $(\alpha_2, \beta_2, \gamma_2)$ will be got by changing x into y in these last equations; $(\alpha_3, \beta_3, \gamma_3)$ by changing x into z, and $(\alpha_4, \beta_4, \gamma_4)$ by changing x into w. To get the cosines $\delta_1, \delta_2, \delta_3, \delta_4$ we need only to recall the properties of the orthogonal substitution for which $\alpha_1, \dots. \delta_4$ are the coefficients. Or we may use the equations which express that the normal whose direction cosines are $\delta_1, \delta_2, \delta_3, \delta_4$ is at right angles to the tangent hyperplane; these are

$$\left.\begin{aligned}
\delta_1 \frac{\partial x}{\partial t} + \delta_2 \frac{\partial y}{\partial t} + \delta_3 \frac{\partial z}{\partial t} + \delta_4 \frac{\partial w}{\partial t} = 0, \\
\delta_1 \frac{\partial x}{\partial u} + \delta_2 \frac{\partial y}{\partial u} + \delta_3 \frac{\partial z}{\partial u} + \delta_4 \frac{\partial w}{\partial u} = 0, \\
\delta_1 \frac{\partial x}{\partial v} + \delta_2 \frac{\partial y}{\partial v} + \delta_3 \frac{\partial z}{\partial v} + \delta_4 \frac{\partial w}{\partial v} = 0.
\end{aligned}\right\} \tag{47}$$

We shall have by either method

$$\left.\begin{aligned}
\delta_1 = \frac{1}{\Delta}\, \frac{\partial (y,\, z,\, w)}{\partial (t,\, u,\, v)}, \quad \delta_2 = -\frac{1}{\Delta}\, \frac{\partial (z,\, w,\, x)}{\partial (t,\, u,\, v)}, \\
\delta_3 = \frac{1}{\Delta}\, \frac{\partial (w,\, x,\, y)}{\partial (t,\, u,\, v)}, \quad \delta_4 = -\frac{1}{\Delta}\, \frac{\partial (x,\, y,\, z)}{\partial (t,\, u,\, v)}.
\end{aligned}\right\} \tag{48}$$

The direction cosines P^k_{ij}, thirty-six in number, of the six coordinate planes are readily found when the 16 cosines α_i, β_i, γ_i, δ_i $(i = 1, 2, 3, 4)$ are known.

To obtain the rotations p_{ij}, p'_{ij}, p''_{ij}, we proceed as on p. 378, t. II of Darboux, using equations (43), (44), (45) and (48). We have at once the following six relations:

$$\left.\begin{aligned}
\sum \delta_1 d \frac{\partial x}{\partial t} = {} & \xi\, (p_{14}dt + p'_{14}du + p''_{14}dv) \\
& + \eta\, (p_{24}dt + p'_{24}du + p''_{24}dv) \\
& + \zeta\, (p_{34}dt + p'_{34}du + p''_{34}dv),
\end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned}
\sum \delta_1 d \frac{\partial x}{\partial u} = {} & \xi'\, (p_{14}dt + p'_{14}du + p''_{14}dv), \\
& + \eta'\, (p_{24}dt + p'_{24}du + p''_{24}dv), \\
& + \zeta'\, (p_{34}dt + p'_{34}du + p''_{34}dv),
\end{aligned}\right\} \tag{50}$$

$$\left.\begin{aligned}
\sum \delta_1 d \frac{\partial x}{\partial v} = {} & \xi''\, (p_{14}dt + p'_{14}du + p''_{14}dv), \\
& + \eta''\, (p_{24}dt + p'_{24}du + p''_{24}dv), \\
& + \zeta''\, (p_{34}dt + p'_{34}du + p''_{34}dv),
\end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned}
\sum \frac{\partial x}{\partial t} d \frac{\partial x}{\partial u} = {} & \xi d\xi' + \eta d\eta' + \zeta d\zeta' + (\xi'\eta - \xi\eta')(p_{12}dt + p'_{12}du + p''_{12}dv) \\
& + (\xi\zeta' - \xi'\zeta)(p_{13}dt + p'_{13}du + p''_{13}dv) \\
& + (\eta'\zeta - \eta\zeta')(p_{23}dt + p'_{23}du + p''_{23}dv),
\end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned}
\sum \frac{\partial x}{\partial u} d \frac{\partial x}{\partial v} = {} & \xi' d\xi'' + \eta' d\eta'' + \zeta' d\zeta'' + (\xi''\eta' - \xi'\eta'')(p_{12}dt + p'_{12}du + p''_{12}dv) \\
& + (\xi'\zeta'' - \xi''\zeta')(p_{13}dt + p'_{13}du + p''_{13}dv) \\
& + (\eta''\zeta' - \eta'\zeta'')(p_{23}dt + p'_{23}du + p''_{23}dv),
\end{aligned}\right\} \tag{53}$$

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial v}\,d\,\frac{\partial x}{\partial t} = \xi''d\zeta + \eta''d\eta + \zeta''d\zeta + (\xi\eta'' - \zeta''\eta)(p_{12}dt + p'_{12}du + p''_{12}dv)\\ + (\xi''\zeta - \xi\zeta'')(p_{13}dt + p'_{13}du + p''_{13}dv)\\ + (\eta\zeta'' - \eta''\zeta)(p_{23}dt + p'_{23}du + p''_{23}dv)\,.\end{aligned}\right\} \quad (54)$$

Define now the six determinants

$$D_{11},\ D_{22},\ D_{33},\ D_{12},\ D_{13},\ D_{23}$$

by the equation

$$\left.\begin{aligned}&D_{11}dt^2 + D_{22}du^2 + D_{33}dv^2 + 2D_{12}dt\,du + 2D_{13}dt\,dv + 2D_{23}du\,dv\\ &\quad = \frac{1}{\Delta}\left|\begin{matrix}\dfrac{\partial x}{\partial t}, & \dfrac{\partial x}{\partial u}, & \dfrac{\partial x}{\partial v}, & \dfrac{\partial^2 x}{\partial t^2}dt^2 + \dfrac{\partial^2 x}{\partial u^2}du^2 + \dfrac{\partial^2 x}{\partial v^2}dv^2\\ & & & + 2\dfrac{\partial^2 x}{\partial t\,\partial u}dt\,du + 2\dfrac{\partial^2 x}{\partial t\,\partial v}dt\,dv + 2\dfrac{\partial^2 x}{\partial u\,\partial v}du\,dv,\end{matrix}\right|\end{aligned}\right\} \quad (55)$$

only one line of the determinant on the right-hand side is written, the others are, of course, obtained by changing x into y, z and w successively.

Introducing the values of δ_1, δ_2, δ_3, δ_4, we have

$$\left.\begin{aligned}D_{11} &= \sum\delta_1\frac{\partial^2 x}{\partial t^2}, & D_{22} &= \sum\delta_1\frac{\partial^2 x}{\partial u^2}, & D_{33} &= \sum\delta_1\frac{\partial^2 x}{\partial v^2},\\ D_{12} &= \sum\delta_1\frac{\partial^2 x}{\partial t\,\partial u}, & D_{13} &= \sum\delta_1\frac{\partial^2 x}{\partial t\,\partial v}, & D_{23} &= \sum\delta_1\frac{\partial^2 x}{\partial u\,\partial v}.\end{aligned}\right\} \quad (56)$$

Equations (44), (50) and (51) can now be written

$$\left.\begin{aligned}D_{11}dt + D_{12}du + D_{13}dv &= \xi P_{14} + \eta P_{24} + \zeta P_{34},\\ D_{12}dt + D_{22}du + D_{23}dv &= \xi' P_{14} + \eta' P_{24} + \zeta' P_{34},\\ D_{13}dt + D_{23}du + D_{33}dv &= \xi'' P_{14} + \eta'' P_{24} + \zeta'' P_{34}.\end{aligned}\right\} \quad (57)$$

The P_{14}, P_{24}, P_{34} are abbreviations whose definitions are obvious. We have now

$$\Delta P_{14} = \begin{vmatrix}D_{11}dt + D_{12}du + D_{13}dv, & \eta & \zeta,\\ D_{12}dt + D_{22}du + D_{23}dv, & \eta' & \zeta',\\ D_{13}dt + D_{23}du + D_{33}dv, & \eta'' & \zeta'',\end{vmatrix} \quad (58)$$

or substituting for P_{14} its value,

$$\left.\begin{aligned}&\Delta\,(p_{14}dt + p'_{14}du + p''_{14}dv) =\\ &\quad\begin{vmatrix}D_{11} & \eta & \zeta\\ D_{12} & \eta' & \zeta'\\ D_{13} & \eta'' & \zeta''\end{vmatrix}dt + \begin{vmatrix}D_{12} & \eta & \zeta\\ D_{22} & \eta' & \zeta'\\ D_{23} & \eta'' & \zeta''\end{vmatrix}du + \begin{vmatrix}D_{13} & \eta & \zeta\\ D_{23} & \eta' & \zeta'\\ D_{33} & \eta'' & \zeta''\end{vmatrix}dv.\end{aligned}\right\} \quad (59)$$

Similarly,

$$\left.\begin{aligned}&\Delta(p_{24}dt + p'_{24}du + p''_{24}dv) = \\ &\qquad\begin{vmatrix} \xi & D_{11} & \zeta \\ \xi' & D_{12} & \zeta' \\ \xi'' & D_{13} & \zeta'' \end{vmatrix} dt + \begin{vmatrix} \xi & D_{12} & \zeta \\ \xi' & D_{22} & \zeta' \\ \xi'' & D_{23} & \zeta'' \end{vmatrix} du + \begin{vmatrix} \xi & D_{13} & \zeta \\ \xi' & D_{23} & \zeta' \\ \xi'' & D_{33} & \zeta'' \end{vmatrix} dv\end{aligned}\right\} \quad (60)$$

and

$$\left.\begin{aligned}&\Delta(p_{34}dt + p'_{34}du + p''_{34}dv) = \\ &\qquad\begin{vmatrix} \xi & \eta & D_{11} \\ \xi' & \eta' & D_{12} \\ \xi'' & \eta'' & D_{13} \end{vmatrix} dt + \begin{vmatrix} \xi & \eta & D_{12} \\ \xi' & \eta' & D_{22} \\ \xi'' & \eta'' & D_{23} \end{vmatrix} du + \begin{vmatrix} \xi & \eta & D_{13} \\ \xi' & \eta' & D_{23} \\ \xi'' & \eta'' & D_{33} \end{vmatrix} dv.\end{aligned}\right\} \quad (61)$$

It is only necessary in each of these last three equations to equate separately the coefficients of dt, du, dv and so obtain the values of the nine rotations

$$\begin{matrix} p_{14}, & p'_{14}, & p''_{14}, \\ p_{24} & p'_{24}, & p''_{24}, \\ p_{34}, & p'_{34}, & p''_{34}. \end{matrix}$$

To get the remaining nine rotations $p_{12}, p'_{13}, \ldots . p''_{23}$ we use equations (52), (53) and (54).

We have first to calculate the values of the left-hand members of these equations.

From (52) we have

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial t}\, d\,\frac{\partial x}{\partial u} &= dt\sum\frac{\partial x}{\partial t}\;\frac{\partial^2 x}{\partial t\,\partial u} + du\sum\frac{\partial x}{\partial t}\;\frac{\partial^2 x}{\partial u^2} + dv\sum\frac{\partial x}{\partial t}\;\frac{\partial^2 x}{\partial u\,\partial v} \\ &= \tfrac{1}{2}\,\frac{\partial E_{11}}{\partial u}\,dt + \left[\frac{\partial F_{12}}{\partial u} - \tfrac{1}{2}\,\frac{\partial E_{22}}{\partial t}\right] du + \tfrac{1}{2}\left[\frac{\partial F_{13}}{\partial u} + \frac{\partial F_{12}}{\partial v} - \frac{\partial F_{23}}{\partial t}\right] dv.\end{aligned}\right\} \quad (62)$$

From (53) we have

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial u}\, d\,\frac{\partial x}{\partial v} &= dt\sum\frac{\partial x}{\partial u}\;\frac{\partial^2 x}{\partial t\,\partial v} + du\sum\frac{\partial x}{\partial u}\;\frac{\partial^2 x}{\partial u\,\partial v} + dv\sum\frac{\partial x}{\partial u}\;\frac{\partial^2 x}{\partial v^2} \\ &= \tfrac{1}{2}\left[\frac{\partial F_{12}}{\partial v} - \frac{\partial F_{13}}{\partial u} + \frac{\partial F_{23}}{\partial t}\right] dt + \tfrac{1}{2}\,\frac{\partial E_{22}}{\partial v}\,du + \left[\frac{\partial F_{13}}{\partial v} - \tfrac{1}{2}\,\frac{\partial E_{33}}{\partial u}\right] dv;\end{aligned}\right\} \quad (63)$$

and from (54) we get

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial v}\, d\,\frac{\partial x}{\partial t} &= dt\sum\frac{\partial x}{\partial v}\;\frac{\partial^2 x}{\partial t^2} + du\sum\frac{\partial x}{\partial v}\;\frac{\partial^2 x}{\partial u\,\partial t} + dv\sum\frac{\partial x}{\partial v}\;\frac{\partial^2 x}{\partial t\,\partial v} \\ &= \left[\frac{\partial F_{13}}{\partial t} - \tfrac{1}{2}\,\frac{\partial E_{11}}{\partial v}\right] dt + \tfrac{1}{2}\left[\frac{\partial F_{13}}{\partial u} - \frac{\partial F_{12}}{\partial v} + \frac{\partial F_{23}}{\partial t}\right] du + \tfrac{1}{2}\,\frac{\partial E_{33}}{\partial t}\,dv.\end{aligned}\right\} \quad (64)$$

For brevity, write these last three equations in the form

$$\sum \frac{\partial x}{\partial t}\, d\frac{\partial x}{\partial u} = A_1 dt + B_1 du + C_1 dv, \tag{62'}$$

$$\sum \frac{\partial x}{\partial u}\, d\frac{\partial x}{\partial v} = A_2 dt + B_2 du + C_2 dv, \tag{63'}$$

$$\sum \frac{\partial x}{\partial v}\, d\frac{\partial x}{\partial t} = A_3 dt + B_3 du + C_3 dv. \tag{64'}$$

The values of the A_i, B_i, C_i are seen at once by comparing the equations (62), (63), (64) with (62′), (63′), (64′).

Write also for brevity

$$\left.\begin{aligned} P_{12} &= p_{12}dt + p'_{12}du + p''_{12}dv, \\ P_{13} &= p_{13}dt + p'_{13}du + p''_{13}dv, \\ P_{23} &= p_{23}dt + p'_{23}du + p''_{23}dv, \end{aligned}\right\} \tag{65}$$

Equations (52), (53) and (54) can now be put in the following forms:

$$\left.\begin{aligned} &\left[A_1 - \xi\frac{\partial \xi'}{\partial t} - \eta\frac{\partial \eta'}{\partial t} - \zeta\frac{\partial \zeta'}{\partial t}\right] dt + \left[B_1 - \xi\frac{\partial \xi'}{\partial u} - \eta\frac{\partial \eta'}{\partial u} - \zeta\frac{\partial \zeta'}{\partial u}\right] du \\ &\qquad + \left[C_1 - \xi\frac{\partial \xi'}{\partial v} - \eta\frac{\partial \eta'}{\partial v} - \zeta\frac{\partial \zeta'}{\partial v}\right] dv \\ &= -(\xi\eta' - \xi'\eta) P_{12} - (\zeta\xi' - \zeta'\xi) P_{13} - (\eta\zeta' - \eta'\zeta) P_{23}, \end{aligned}\right\} \tag{52'}$$

$$\left.\begin{aligned} &\left[A_2 - \xi'\frac{\partial \xi''}{\partial t} - \eta'\frac{\partial \eta''}{\partial t} - \zeta'\frac{\partial \zeta''}{\partial t}\right] dt + \left[B_2 - \xi'\frac{\partial \xi''}{\partial u} - \eta'\frac{\partial \eta''}{\partial u} - \zeta'\frac{\partial \zeta''}{\partial u}\right] du \\ &\qquad + \left[C_2 - \xi'\frac{\partial \xi''}{\partial v} - \eta'\frac{\partial \eta''}{\partial v} - \zeta'\frac{\partial \zeta''}{\partial v}\right] dv \\ &= -(\xi'\eta'' - \xi''\eta') P_{12} - (\xi''\zeta' - \xi'\zeta'') P_{13} - (\eta'\zeta'' - \eta''\zeta') P_{23}, \end{aligned}\right\} \tag{53'}$$

$$\left.\begin{aligned} &\left[A_3 - \xi''\frac{\partial \xi}{\partial t} - \eta''\frac{\partial \eta}{\partial t} - \zeta''\frac{\partial \zeta}{\partial t}\right] dt + \left[B_3 - \xi''\frac{\partial \xi}{\partial u} - \eta''\frac{\partial \eta}{\partial u} - \zeta''\frac{\partial \zeta}{\partial u}\right] du \\ &\qquad + \left[C_3 - \xi''\frac{\partial \xi}{\partial v} - \eta''\frac{\partial \eta}{\partial v} - \zeta''\frac{\partial \zeta}{\partial v}\right] dv \\ &= -(\xi''\eta - \xi\eta'') P_{12} - (\xi\zeta'' - \xi''\zeta) P_{13} - \eta''\zeta - (\eta''\zeta - \eta\zeta'') P_{23}. \end{aligned}\right\} \tag{54'}$$

The determinant of the right-hand sides of these equations is obviously

$$-\Delta^2 = -\begin{vmatrix} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \xi'', & \eta'', & \zeta'' \end{vmatrix}^2 = -\begin{vmatrix} E_{11}, & F_{12}, & F_{13}, \\ F_{12}, & E_{22}, & F_{23}, \\ F_{13}, & F_{23}, & E_{33}. \end{vmatrix}$$

We have now only to solve these last three equations in order to obtain the values of P_{12}, P_{13}, P_{23}, then replacing these quantities by their values from (65) and equating separately the coefficients of dt, du, dv on each side of the results, we shall have the values of the nine rotations

$$\begin{matrix} p_{12}, & p'_{12}, & p''_{12}, \\ p_{13}, & p'_{13}, & p''_{13}, \\ p_{23}, & p'_{23}, & p''_{23}, \end{matrix}$$

which, it will be noted, depend only on the linear element. The work of calculation is somewhat long, but perfectly simple. We find

$$\left.\begin{aligned} P_{12} &= p_{12}dt + p'_{12}du + p''_{12}dv \\ &= -\frac{1}{\Delta}\Big\{\xi''\Big[\Big(A_1-\sum\xi\frac{\partial\xi'}{\partial t}\Big)dt+\Big(B_1-\sum\xi\frac{\partial\xi'}{\partial u}\Big)du+\Big(C_1-\sum\xi\frac{\partial\xi'}{\partial v}\Big)dv\Big] \\ &\qquad +\xi\Big[\Big(A_2-\sum\xi'\frac{\partial\xi''}{\partial t}\Big)dt+\Big(B_2-\sum\xi'\frac{\partial\xi''}{\partial u}\Big)du+\Big(C_2-\sum\xi'\frac{\partial\xi''}{\partial v}\Big)dv\Big] \\ &\qquad +\xi\Big[\Big(A_3-\sum\xi''\frac{\partial\xi}{\partial t}\Big)dt+\Big(B_3-\sum\xi''\frac{\partial\xi}{\partial u}\Big)du+\Big(C_3-\sum\xi''\frac{\partial\xi}{\partial v}\Big)dv\Big]\Big\} \end{aligned}\right\}\tag{66}$$

We shall have P_{13} and P_{23} by changing the multipliers ξ'', ξ, ξ' of the brackets into η'', η, η' and ζ'', ζ, ζ' respectively. If we take the simplest case, viz. the one corresponding to lines of curvature on a surface in three-dimensional space, we shall have

$$\left.\begin{aligned} D_{12} = D_{13} = D_{23} = 0\,, \\ F_{12} = F_{13} = F_{23} = 0\,. \end{aligned}\right\}\tag{67}$$

It is easy to see that these last conditions involve the following:

$$\xi' = \xi'' = \eta = \eta'' = \zeta = \zeta' = 0,$$

and so

$$\left.\begin{aligned} &\xi = \sqrt{E_{11}}, \quad \eta' = \sqrt{E_{22}}, \quad \zeta'' = \sqrt{E_{33}}, \\ &\Delta = \xi\eta'\zeta'' = \sqrt{E_{11}E_{22}E_{33}}. \end{aligned}\right\}\tag{69}$$

From equations (59), (60) and (61) we now get

$$\left.\begin{matrix} p_{14} = \dfrac{D_{11}}{\sqrt{E_{11}}}, & p'_{14} = 0 \quad, & p''_{14} = 0 \quad, \\ p_{24} = 0 \quad, & p'_{24} = \dfrac{D_{22}}{\sqrt{E_{22}}}, & p''_{24} = 0 \quad, \\ p_{34} = 0 \quad, & p'_{34} = 0 \quad, & p''_{34} = \dfrac{D_{33}}{\sqrt{E_{33}}}\,. \end{matrix}\right\}\tag{70}$$

These give us, of course, expressions for the three principal radii of curvature at a point of the hypersurface where the three parametric surfaces

$$t = \text{const.}, \quad u = \text{const.}, \quad v = \text{const.}$$

are *surfaces of curvature.*

We have also from (66) and the two similar equations derived from it by changing ξ into η and into ζ respectively:

$$\left.\begin{array}{lll}
p_{12} = 0 \quad , & p'_{12} = \dfrac{-1}{2\sqrt{E_{22}E_{33}}}\dfrac{\partial E_{22}}{\partial v}, & p''_{12} = \dfrac{1}{2\sqrt{E_{22}E_{33}}}\dfrac{\partial E_{33}}{\partial u}, \\
p_{13} = \dfrac{1}{2\sqrt{E_{11}E_{33}}}\dfrac{\partial E_{11}}{\partial v}, & p'_{13} = 0 \quad , & p''_{13} = \dfrac{-1}{2\sqrt{E_{11}E_{33}}}\dfrac{\partial E_{33}}{\partial t}, \\
p_{23} = \dfrac{-1}{2\sqrt{E_{11}E_{22}}}\dfrac{\partial E_{11}}{\partial u}, & p'_{23} = \dfrac{1}{2\sqrt{E_{11}E_{22}}}\dfrac{\partial E_{12}}{\partial t}, & p''_{23} = 0
\end{array}\right\} \quad (71)$$

These formulæ are analogous to those for the radii of geodesic curvature of the lines of curvature in ordinary space.

It is obvious that an enormous number of formulæ concerning hypersurfaces, surfaces and curves in a four-dimensional space, can be derived from the preceding brief generalization of Darboux's methods. A problem of particular interest, as it seems to me, is the case where the three variables t, u, v depend on two independent parameters; for the study of this problem Poincaré's memoir ("Sur les Résidus des Intégrales doubles," Acta Math., t. IX) and the treatise "Théorie des Fonctions Algébriques de deux Variables indépendantes," by Picard and Simart are full of suggestions.

BALTIMORE, *Dec.* 2, 1897.